1 Problem

Equation 1 models the motion of a double pendulum for small oscillations about equilibrium with a perpendicular force acting at the end of the lower pendulum.

$$\begin{bmatrix} (m_1+m_2){l_1}^2 & m_2 l_2 l_1 \\ m_2 l_1 l_2 & m_2 {l_2}^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (w_1+w_2)l_1 & 0 \\ 0 & w_2 l_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} f l_1 \\ f l_2 \end{Bmatrix} \tag{1}$$

For $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, and $l_1 = l_2 = 0.5\,\text{m}$ perform the following engineering analysis tasks:

a. State the characteristic equation.

b. Calculate the eigenvalues and natural frequencies.

c. Calculate teh matrix of eigenvectors $[A]$.

d. Normalize the eigenvectors such that $[A^*]^T[M][A^*] = [I]$ and comfirm that your normalized eigenvectors satify $[A^*]^T[K][A^*] = [\Lambda]$, where $[\Lambda]$ is a diagonal matrix with the eigenvalues as diagon elements.

e. State the modal differential equations (including the modal force definitions) and the coordiante transformation from the modal coordinate model to physical coordinates.

2 Solution

a. You start by substituting in the values given for m_1, m_2, l_1, and l_2 (because solving for eigenvalues symbolically would be a real pain). The values for w_n is simply $m_n g$, in this case $g = 9.81\,\text{m/s}^2$ because we are working in the metric system. When you perform this substitution we get the following:

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = f\{l\}$$

where:

$$[M] = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

and:

$$[K] = \begin{bmatrix} 14.715 & 0 \\ 0 & 9.81 \end{bmatrix}$$

The characteristic equation is shown in Equation 2.

$$|-\lambda\,[M] + [K]| = 0 \tag{2}$$

Which, after some substitution is:

$$\begin{vmatrix} 14.715 - 0.75\lambda & -0.5\lambda \\ -0.5\lambda & 9.81 - 0.5\lambda \end{vmatrix} = 0$$

Using the definition of a determinant (which is denoted by the vertical bar, or pipe) we get:

$$0.125\lambda^2 - 14.715\lambda + 144.354 = 0$$

b. Solving for λ yields: $\lambda_1 = 10.801$, and $\lambda_2 = 106.919$ (these should be ordered from lowest to highest), and since $\lambda \equiv \omega_n^2$ $\omega_{n_1} = 3.2865$, and $\omega_{n_2} = 10.3402$. Therefore the eigenvalues are:

$$\begin{bmatrix} 10.801 \\ 106.919 \end{bmatrix}$$

c. Apply Equation 3 for each value of λ while setting $a_1 = 1$ and find a_2.

$$(-\lambda\,[M] + [K])\,\{a\} = 0 \tag{3}$$

Which, of course is:

$$\begin{bmatrix} 14.715 - 0.75\lambda & -0.5\lambda \\ -0.5\lambda & 9.81 - 0.5\lambda \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

Solving this with λ_1 gives:

$$6.614 - 5.401a_2 = 0, \text{ or } a_2 = 1.225$$

Plugging in λ_2 gives:

$$-65.464 - 53.460a_2 = 0, \text{ or} a_2 = -1.225$$

Therefore the eigenvectors are:

$$[A] = \begin{bmatrix} 1 & 1 \\ 1.225 & -1.225 \end{bmatrix}$$

d. The next step is to find $[M_Q]$ which you do by applying Equation 4.

$$[M_Q] = [A]^T[M][A] \tag{4}$$

Which evaluates to:

$$\begin{bmatrix} 1 & 1.225 \\ 1 & -1.225 \end{bmatrix} \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.225 & -1.225 \end{bmatrix} = [M_Q] = \begin{bmatrix} 2.725 & 0 \\ 0 & 0.275 \end{bmatrix}$$

Thus, $M_{Q_11} = 2.725$, and $M_{Q_22} = 0.275$. Equation 5 shows how to find the normalized eigenvectors:

$$[A^*] = \begin{bmatrix} \frac{a_{11}}{\sqrt{M_{Q_{11}}}} & \frac{a_{12}}{\sqrt{M_{Q_{22}}}} \\ \frac{a_{21}}{\sqrt{M_{Q_{11}}}} & \frac{a_{22}}{\sqrt{M_{Q_{22}}}} \end{bmatrix} \tag{5}$$

Therefore:

$$[A^*] = \begin{bmatrix} 0.606 & 1.906 \\ 0.742 & -2.334 \end{bmatrix}$$

To verify $[A^*]$ we use: $[A^*]^T[M][A^*] = [I]$, and $[A^*]^T[K][A^*] = [\Lambda]$

$$\begin{bmatrix} 0.606 & 0.742 \\ 1.906 & -2.334 \end{bmatrix} \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.606 & 1.906 \\ 0.742 & -2.334 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0.606 & 0.742 \\ 1.906 & -2.334 \end{bmatrix} \begin{bmatrix} 14.715 & 0 \\ 0 & 9.81 \end{bmatrix} \begin{bmatrix} 0.606 & 1.906 \\ 0.742 & -2.334 \end{bmatrix} = \begin{bmatrix} 10.801 & 0 \\ 0 & 106.919 \end{bmatrix}$$

e. In order to state the modal differential equations we must find the modal force definitions with Equation 6

$$[Q] = [A^*]^T f\{l\} \tag{6}$$

Which equals:

$$[Q] = \begin{bmatrix} 0.606 & 0.742 \\ 1.906 & -2.334 \end{bmatrix} \begin{Bmatrix} 0.5f \\ 0.5f \end{Bmatrix} = \begin{Bmatrix} 0.674f \\ -0.214f \end{Bmatrix}$$

Thus the modal differential equations are:

$$\ddot{q}_1 + 10.801q_1 = 0.674f, \text{ and } \ddot{q}_2 + 106.919q_2 = -0.214f$$

And to get back to the physical coordinates use Equation 7 after solving for q_1, and q_2.

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = [A^*] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{7}$$